11. Quantum Fields

The creation and destruction operators introduced in the previous chapter almost always occur in the context of quantum fields, even though these fields may have quite variable interpretations. For example, the electromagnetic field can be quantized by use of the canonical quantization picture, and this leads naturally to the interpretation of the quantized amplitude of a field mode in terms of a destruction or creation operator. On the other hand, the description of conserved particles (such as atoms) as Bosons or Fermions leads to the construction of a corresponding quantized matter wave field operator, which obeys the Schrödinger equation. The result is that matter-wave duality becomes universally enmeshed with the concept of field quantization.

11.1 Kinds of Quantum Field

Any field, either classical and quantum, can be expressed as a linear combination of linearly independent *mode functions*. These mode functions can in principle be quite arbitrary, but the most useful modes are the *plane wave* modes. In terms of these plane wave modes, the universal concepts shared by all quantized fields are:

i) The association of each mode, labelled by an index i, with a wavevector $\boldsymbol{k}_i$.

ii) The mode function for each $\boldsymbol{k}_i$.

iii) The expression of the field operator in terms of the creation and destruction operators, and the mode functions.

iv) The choice of Hamiltonian.

In the following we shall formulate the quantization procedure for some of the most common quantum fields. For all of these, we will use box normalization, in which the fields are supposed to be confined within a box of volume V, which is considered to be very large.

11.1.1 Matter Wave Fields

We will consider a spinless particle of mass m, whose modes can be completely characterized by the momentum vector $\hbar\boldsymbol{k}$. Thus for the matter wave field, the mode functions are

$$u_{\boldsymbol{k}}(\boldsymbol{x}) = \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{11.1.1}$$

a) Bosons: These are described in terms of a field operator in the Schrödinger picture, which we will write in the form

$$\psi(\boldsymbol{x}) = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}} u_{\boldsymbol{k}}(\boldsymbol{x}). \tag{11.1.2}$$

b) Commutation Relations for Boson Fields: As a result of the creation and destruction operator commutation relations (as in (10.1.1)), we can derive the field operator commutation relations

$$[\psi(\boldsymbol{x}), \psi^\dagger(\boldsymbol{x}')] = \delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{11.1.3}$$

$$[\psi(\boldsymbol{x}), \psi(\boldsymbol{x}')] = [\psi^\dagger(\boldsymbol{x}), \psi^\dagger(\boldsymbol{x}')] = 0. \tag{11.1.4}$$

These can be interpreted as meaning that the field operator $\psi^\dagger(\boldsymbol{x})$ creates a Bose particle at the point $\boldsymbol{x}$.

c) Hamiltonian: We will use the notation for the mode frequency

$$\hbar\omega_k \equiv \frac{\hbar^2 \boldsymbol{k}^2}{2m}, \tag{11.1.5}$$

and the Hamiltonian becomes

$$H_{\text{matter}} = \sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}, \tag{11.1.6}$$

$$= \int d^3\boldsymbol{x}\, \psi^\dagger(\boldsymbol{x}) \left(-\frac{\hbar^2\nabla^2}{2m}\right) \psi(x). \tag{11.1.7}$$

d) Field Operators in the Heisenberg Picture: In the Heisenberg picture the field operator takes the form

$$\psi(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}} u_{\boldsymbol{k}}(\boldsymbol{x}) e^{-i\omega_{\boldsymbol{k}} t}. \tag{11.1.8}$$

In this equation, the $a_{\boldsymbol{k}}$ are the Schrödinger picture operators. The time dependence is entirely represented in the arguments of the exponentials.

e) Equation of Motion: Using the commutation relations (11.1.3, 11.1.4), one can derive the equation of motion for the quantum field, which is identical in form to the free particle Schrödinger equation:

$$i\hbar \frac{\partial \psi(\boldsymbol{x}, t)}{\partial t} = -\frac{\hbar^2\nabla^2 \psi(\boldsymbol{x}, t)}{2m}. \tag{11.1.9}$$

f) Fermions: The expression of Fermi fields in terms of mode functions is exactly the same as for Bose fields, apart from the operator substitution $a_{\boldsymbol{k}} \to b_{\boldsymbol{k}}$ where $b_{\boldsymbol{k}}$ are Fermi operators. These satisfy, as in (10.1.8), *anticommutation* relations, which lead to the field anticommutation relations

$$[\psi(\boldsymbol{x}), \psi^\dagger(\boldsymbol{x}')]_+ = \delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{11.1.10}$$

$$[\psi(\boldsymbol{x}), \psi(\boldsymbol{x}')]_+ = [\psi^\dagger(\boldsymbol{x}), \psi^\dagger(\boldsymbol{x}')]_+ = 0. \tag{11.1.11}$$

The field operator $\psi^\dagger(\boldsymbol{x})$ can be taken as the creation operator of a Fermi particle at $\boldsymbol{x}$. Even though the commutation relations have been replaced by anticommutation relations, the Hamiltonian and equations of motion take exactly the same form as for the Boson field, namely (11.1.7) and (11.1.9).

11.1.2 Sound Waves

Sound waves occur in a range of media—solid, liquid or gaseous—so here we will simply give a generic and simplified description of their formulation and quantization. We introduce a variable $f(\boldsymbol{x},t)$ which represents the deviation of the medium from its equilibrium state. For example, for sound waves in an elastic solid, this is a measure of the strain, while for a gas it might be the deviation of the pressure or the density from equilibrium. We take the resulting potential and kinetic energy in the forms

$$E_{\rm pot} \equiv \frac{1}{2\kappa}\int d^3\boldsymbol{x}\,(\nabla f)^2, \tag{11.1.12}$$

$$E_{\rm kin} \equiv \frac{\lambda}{2}\int d^3\boldsymbol{x}\left(\frac{\partial f}{\partial t}\right)^2. \tag{11.1.13}$$

Here we have introduced κ, which is a measure of elasticity, and λ, which is a measure of inertia.

The wave equation resulting from these can be deduced using Lagrange's equations, and is

$$\frac{\partial^2 f}{\partial t^2} = c_{\rm s}^2\nabla^2 f, \tag{11.1.14}$$

with the speed of sound $c_{\rm s}$ being given by

$$c_{\rm s} = \frac{1}{\sqrt{\kappa\lambda}}. \tag{11.1.15}$$

a) Mode Functions: We now introduce mode functions and frequencies

$$u_{\boldsymbol{k}}(\boldsymbol{x}) \equiv \frac{1}{\sqrt{V}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{11.1.16}$$

$$\omega_{\boldsymbol{k}} = |\boldsymbol{k}|c_{\rm s}, \tag{11.1.17}$$

and write the operator expansion

$$f(\boldsymbol{x},t) = f^{(+)}(\boldsymbol{x},t) + f^{(-)}(\boldsymbol{x},t), \tag{11.1.18}$$

where $f^{(+)}(\boldsymbol{x},t)$ is called the *positive frequency* part of the sound field, and the corresponding *negative frequency* part is $f^{(-)}(\boldsymbol{x},t)$. These are defined by

$$f^{(+)}(\boldsymbol{x},t) \equiv \sum_{\boldsymbol{k}}\sqrt{\frac{\hbar}{2\lambda\omega_{\boldsymbol{k}}}}\;a_{\boldsymbol{k}}u_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}}t}, \tag{11.1.19}$$

$$f^{(-)}(\boldsymbol{x},t) \equiv \sum_{\boldsymbol{k}}\sqrt{\frac{\hbar}{2\lambda\omega_{\boldsymbol{k}}}}\;a_{\boldsymbol{k}}^\dagger u^*(\boldsymbol{x})_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}}t}. \tag{11.1.20}$$

By construction, the frequency-wavenumber relation (11.1.17) assures that $f(\boldsymbol{x},t)$ and $f^{(\pm)}(\boldsymbol{x},t)$ are all solutions of the wave equation (11.1.14). The wave equation is to be interpreted as a Heisenberg equation of motion for the field operator $f(\boldsymbol{x},t)$.

b) Commutation Relations: We interpret $a_{\boldsymbol{k}}^{\dagger}$ and $a_{\boldsymbol{k}}$ as creation and destruction operators, and introduce the commutation relations

$$\left[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^{\dagger}\right] = \delta_{\boldsymbol{k},\boldsymbol{k}'}. \tag{11.1.21}$$

From these we can deduce the field commutation relations

$$\left[\frac{\partial f(\boldsymbol{x},t)}{\partial t}, f(\boldsymbol{x}',t)\right] = \frac{i\hbar}{\lambda}\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{11.1.22}$$

These commutation relations can also be obtained using an appropriate Lagrangian to derive the canonical co-ordinates and momenta, which are then quantized by imposing canonical commutation relations.

c) Hamiltonian: The Hamiltonian is obtained from the kinetic and potential energies (11.1.12, 11.1.13), and takes the form

$$H_{\text{sound}} = \tfrac{1}{2}\int d^3\boldsymbol{x}\left\{\frac{(\nabla f)^2}{\kappa} + \lambda\left(\frac{\partial f}{\partial t}\right)^2\right\}, \tag{11.1.23}$$

$$= \sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}}\left(a_{\boldsymbol{k}}^{\dagger}a_{\boldsymbol{k}} + \tfrac{1}{2}\right). \tag{11.1.24}$$

d) Equation of Motion: Using the Hamiltonian and the commutation relations, the Heisenberg equations of motion can be derived for both $f(\boldsymbol{x},t)$ and its time derivative $\partial f(\boldsymbol{x},t)/\partial t$, which behaves as the canonical momentum conjugate to $f(\boldsymbol{x},t)$, because of the commutation relation (11.1.22) between them. These yield (11.1.14), the sound wave equation.

e) Comparison to Matter Wave Field Operators: The sound field $f(\boldsymbol{x},t)$ contains the creation and destruction operators in equal proportions, while the matter wave fields $\psi(\boldsymbol{x},t)$, $\psi^{\dagger}(\boldsymbol{x},t)$ consists only of either destruction or creation operators. The explicit inclusion of the factor $1/\sqrt{\omega_{\boldsymbol{k}}}$ means that there is no simple commutation relation between $f^{(+)}$ and $f^{(-)}$. However, this factor is also the reason that $f(\boldsymbol{x},t)$, and $\partial f(\boldsymbol{x},t)/\partial t$ obey the canonical commutation relation (11.1.22), since it cancels with the factor of $\omega_{\boldsymbol{k}}$ which arises from the time derivative.

11.1.3 The Electromagnetic Field

The electromagnetic field equations in free space (that is, with no charge or current sources, and no dielectric or permeable materials) are the quantized versions of Maxwell's equations. Their quantization is similar to that for sound waves, but is considerably more technically complex because of the vector nature of the fields involved.

a) Heisenberg Equations of Motion: In this case it is simpler to work in the Heisenberg picture, and for the electromagnetic field operators the Heisenberg equations of motion are Maxwell's equations

$$\nabla\cdot\boldsymbol{D} = 0, \tag{11.1.25}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{11.1.26}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \tag{11.1.27}$$

$$\nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t}, \tag{11.1.28}$$

where $\boldsymbol{B} = \mu_0\boldsymbol{H}, \boldsymbol{D} = \varepsilon_0\boldsymbol{E}$, where μ_0, ε_0 are the magnetic permeability and electric permittivity of free space and $\mu_0\varepsilon_0 = c^{-2}$.

b) Potentials: The electromagnetic field is best represented in terms of the vector and scalar potentials $\boldsymbol{A}$ and ϕ, in terms of which

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{11.1.29}$$

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}. \tag{11.1.30}$$

However there is no unique set of $\boldsymbol{A}$ and ϕ which specify a given $\boldsymbol{B}$ and $\boldsymbol{E}$ since a *gauge transformation*

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi, \tag{11.1.31}$$

$$\phi' = \phi - \frac{\partial\chi}{\partial t}, \tag{11.1.32}$$

does not change the measurable fields $\boldsymbol{B}$ and $\boldsymbol{E}$.

c) Coulomb Gauge: For the purposes of optics the *Coulomb gauge* is convenient; this is defined by the choice of a *time-independent* ϕ, and a transverse vector potential, thus

$$\phi(\boldsymbol{x}, t) \to \phi(\boldsymbol{x}), \tag{11.1.33}$$

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{x}, t) = 0. \tag{11.1.34}$$

Thus, we can write

$$\boldsymbol{B}(\boldsymbol{x}, t) = \nabla\times\boldsymbol{A}(\boldsymbol{x}, t), \tag{11.1.35}$$

$$\boldsymbol{E}(\boldsymbol{x}, t) = -\nabla\phi(\boldsymbol{x}) - \frac{\partial\boldsymbol{A}(\boldsymbol{x}, t)}{\partial t}, \tag{11.1.36}$$

with the transversality or Coulomb gauge condition (11.1.34).

This particular choice of gauge is not the only one—relativistically invariant choices are also possible, and are much more appropriate for more advanced work.

d) Wave Equation: The expansion in modes is essentially the same kind of expansion used for the one component sound field of Sect. 11.1.2, with some complications introduced by the vector nature of the fields, and the transversality condition. Substituting (11.1.35) in (11.1.27) we find that $\boldsymbol{A}(\boldsymbol{x},t)$ satisfies the wave equation

$$\nabla^2 \boldsymbol{A}(\boldsymbol{x},t) = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}(\boldsymbol{x},t)}{\partial t^2}. \tag{11.1.37}$$

Maxwell's equations, and the wave equation (11.1.37) for the the vector potential, will be the Heisenberg equations of motion which should arise from the correct choice of Hamiltonian operator. The electrostatic potential $\phi(\boldsymbol{x})$ on the other hand remains as an unquantized c-number.

e) Expansion in Mode Functions: As for sound waves, we separate the vector potential into positive and negative frequency terms

$$\boldsymbol{A}(\boldsymbol{x},t) = \boldsymbol{A}^{(+)}(\boldsymbol{x},t) + \boldsymbol{A}^{(-)}(\boldsymbol{x},t). \tag{11.1.38}$$

Here $\boldsymbol{A}^{(+)}(\boldsymbol{x},t)$ contains only Fourier components with positive frequency, i.e., only terms which vary as $\mathrm{e}^{-\mathrm{i}\omega t}$ for $\omega > 0$, and $\boldsymbol{A}^{(-)}(\boldsymbol{x},t)$ contains amplitudes which vary as $\mathrm{e}^{\mathrm{i}\omega t}$. We take $\boldsymbol{A}$ to be Hermitian so $\boldsymbol{A}^{(-)}(\boldsymbol{x},t) = \{\boldsymbol{A}^{(+)}(\boldsymbol{x},t)\}^\dagger$.

f) Mode Functions: The positive frequency part of vector potential is expanded in terms of the discrete set of orthogonal mode functions and destruction operators as

$$\boldsymbol{A}^{(+)}(\boldsymbol{x},t) = \sum_k a_k \boldsymbol{u}_k(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega_k t}. \tag{11.1.39}$$

The set of vector mode functions $\boldsymbol{u}_k(\boldsymbol{x})$ which correspond to frequency ω_k will satisfy the wave equation

$$\left(\nabla^2 + \frac{\omega_k^2}{c^2}\right)\boldsymbol{u}_k(\boldsymbol{x}) = 0. \tag{11.1.40}$$

The mode functions are also required to satisfy the transversality condition which arises from (11.1.34)

$$\nabla\cdot\boldsymbol{u}_k(\boldsymbol{x}) = 0. \tag{11.1.41}$$

They also form an orthonormal set

$$\int_V \boldsymbol{u}_k^*(\boldsymbol{x})\cdot\boldsymbol{u}_{k'}(\boldsymbol{x})\,d^3\boldsymbol{x} = \delta_{kk'}, \tag{11.1.42}$$

which is complete within the chosen volume. Plane wave mode functions may be written as

$$\boldsymbol{u}_k(\boldsymbol{x}) = \frac{1}{\sqrt{V}}\hat{\boldsymbol{e}}^{(\lambda)}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}), \tag{11.1.43}$$

where $\hat{\boldsymbol{e}}^{(\lambda)}$ are the unit polarization vectors satisfying

$$\boldsymbol{k}\cdot\hat{\boldsymbol{e}}^{(\lambda)} = 0, \qquad \hat{\boldsymbol{e}}^{(\lambda)*}\cdot\hat{\boldsymbol{e}}^{(\lambda')} = \delta_{\lambda\lambda'}. \tag{11.1.44}$$

(For plane polarization $\hat{\boldsymbol{e}}^{(\lambda)}$ can be chosen real, but this is not possible for circular polarization.) The mode index k describes several discrete variables, the polarization index ($\lambda = 1,2$), and the three Cartesian components of the propagation vector $\boldsymbol{k}$. Thus, we have the equivalence

$$k \longleftrightarrow (\boldsymbol{k}, \lambda),$$

and as a consequence of the wave equation (11.1.40)

$$\omega_k = c|\boldsymbol{k}|. \tag{11.1.45}$$

The polarization vector $\hat{\boldsymbol{e}}^{(\lambda)}$ is required to be perpendicular to $\boldsymbol{k}$ by the transversality condition (11.1.41).

g) Expansion of Field Operators: The vector potential operator may now be written in the form

$$\boldsymbol{A}(\boldsymbol{x}, t) = \sum_k \sqrt{\frac{\hbar}{2\omega_k \varepsilon_0}} \left(a_k \boldsymbol{u}_k(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i}\omega_k t} + a_k^\dagger \boldsymbol{u}_k^*(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\omega_k t} \right). \tag{11.1.46}$$

The corresponding form for the quantized electric field (i.e., excluding the part $-\nabla\phi(\boldsymbol{x})$ arising from the static scalar potential) is

$$\boldsymbol{E}_{\mathrm{rad}}(\boldsymbol{x}, t) = -\frac{\partial \boldsymbol{A}(x, t)}{\partial t}, \tag{11.1.47}$$

$$= \mathrm{i} \sum_k \sqrt{\frac{\hbar \omega_k}{2\varepsilon_0}} \left(a_k \boldsymbol{u}_k(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i}\omega_k t} - a_k^\dagger \boldsymbol{u}_k^*(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\omega_k t} \right). \tag{11.1.48}$$

Thus $\boldsymbol{E}_{\mathrm{rad}}(\boldsymbol{x}, t)$ acts as a variable canonically conjugate to the vector potential $\boldsymbol{A}(\boldsymbol{x}, t)$. The commutation relation does not take a simple delta function form, however, because of the transversality condition.

h) Hamiltonian: The Hamiltonian for the electromagnetic field is given by

$$H_{\mathrm{EM}} = \int \left(\frac{\varepsilon_0 \boldsymbol{E}^2}{2} + \frac{\boldsymbol{B}^2}{2\mu_0} \right) d^3\boldsymbol{x}. \tag{11.1.49}$$

By substituting for $\boldsymbol{E}$ and $\boldsymbol{B}$ using the expression (11.1.48) for $\boldsymbol{E}$ and a similar one for $\boldsymbol{B}$, and by making use of the conditions (11.1.40, 11.1.41), this Hamiltonian can be reduced to the form

$$H_{\mathrm{EM}} = \sum_{\boldsymbol{k},\lambda} c|\boldsymbol{k}| \left(a_{\boldsymbol{k},\lambda}^\dagger a_{\boldsymbol{k},\lambda} + \tfrac{1}{2} \right). \tag{11.1.50}$$

11.1.4 Monochromatic Electromagnetic Waves

No wave is truly monochromatic, so what we mean here is a wave in which the frequencies of interest are around a value Ω, so that we can write an expression for the fields in the form

$$\boldsymbol{A}(\boldsymbol{x}, t) = \left(\frac{\hbar}{2\Omega\varepsilon_0} \right)^{1/2} \left(\mathrm{e}^{-\mathrm{i}\Omega t} \boldsymbol{\Psi}(\boldsymbol{x}, t) + \mathrm{e}^{\mathrm{i}\Omega t} \boldsymbol{\Psi}^\dagger(\boldsymbol{x}, t) \right), \tag{11.1.51}$$

$$\boldsymbol{E}(\boldsymbol{x}, t) = \mathrm{i} \left(\frac{\hbar\Omega}{2\varepsilon_0} \right)^{1/2} \left(\mathrm{e}^{-\mathrm{i}\Omega t} \boldsymbol{\Psi}(\boldsymbol{x}, t) - \mathrm{e}^{\mathrm{i}\Omega t} \boldsymbol{\Psi}^\dagger(\boldsymbol{x}, t) \right), \tag{11.1.52}$$

in which

$$\mathbf{\Psi}(\boldsymbol{x},t) \equiv \sum_k a_k \boldsymbol{u}_k(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i}\tilde{\omega}_k t}, \tag{11.1.53}$$

$$\tilde{\omega}_{\boldsymbol{k}} \equiv \omega_{\boldsymbol{k}} - \Omega. \tag{11.1.54}$$

The intensity of the electromagnetic field is given by the energy density, as expressed in (11.1.49). In the case of a monochromatic field this will contain terms proportional to $\exp(\pm 2\mathrm{i}\Omega t)$, which oscillate so rapidly compared to the other terms that they are not measurable by any ordinary detector, which must average over periods far longer than that of a single optical cycle. The end result is that the operator for the intensity of the detectable electromagnetic field is the rate of photon counting which is determined by the mean value of the intensity operator

$$I(\boldsymbol{x},t) = \left(\frac{\hbar\Omega}{2\varepsilon_0}\right)\mathbf{\Psi}^\dagger(\boldsymbol{x},t)\cdot\mathbf{\Psi}(\boldsymbol{x},t) + \left(\frac{\hbar c^2}{2\Omega}\right)\left(\nabla\times\mathbf{\Psi}^\dagger(\boldsymbol{x},t)\right)\cdot\left(\nabla\times\mathbf{\Psi}(\boldsymbol{x},t)\right). \tag{11.1.55}$$

The vector nature of the field makes for the somewhat complicated formula, but qualitatively, this is very similar to the number density operator for a matter field, $\psi^\dagger(\boldsymbol{x},t)\psi(\boldsymbol{x},t)$.

11.1.5 States of Quantized Fields

The dynamical states of a quantized field may be described by assigning an appropriate quantum state to each of the modes—these modes may be assigned and described independently. All of the apparatus developed for Boson or Fermion operators in Chap. 10 can then be applied to the specification of the relevant quantum states.

In this section we will consider the electromagnetic field only—the results for other fields are very similar.

a) Vacuum State: This is the state $|0\rangle$ with no photons in it, i.e., such that

$$a_k|0\rangle = 0 \text{ for all } k. \tag{11.1.56}$$

In this state, although

$$\langle \boldsymbol{E}(\boldsymbol{x},t)\rangle = \langle 0|\boldsymbol{E}(\boldsymbol{x},t)|0\rangle = 0, \tag{11.1.57}$$

the correlation functions are not zero, in fact

$$\langle 0|\boldsymbol{E}(\boldsymbol{x},t)\boldsymbol{E}(\boldsymbol{x}',t')|0\rangle = \sum_{k,k'}\left(\frac{\hbar^2\omega_{k'}\omega_k}{4\varepsilon_0^2}\right)^{\frac{1}{2}} \boldsymbol{u}_k(\boldsymbol{x})\boldsymbol{u}^*_{k'}(\boldsymbol{x}')\mathrm{e}^{-\mathrm{i}(\omega_k t - \omega_{k'}t')}\delta_{k,k'}, \tag{11.1.58}$$

$$= \sum_k \left(\frac{\hbar\omega_k}{2\varepsilon_0 L^3}\right)\boldsymbol{e}^{\lambda *}_{\boldsymbol{k}}\,\boldsymbol{e}^{\lambda}_{\boldsymbol{k}}\mathrm{e}^{-\mathrm{i}\omega_k(t-t')+\mathrm{i}\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}. \tag{11.1.59}$$

This represents *vacuum fluctuations*, in which even in the vacuum there is a nonvanishing fluctuating electromagnetic field, whose average is zero.

b) Coherent State: If every mode of the electromagnetic field is in a *coherent state* $|\boldsymbol{\alpha}\rangle$, such that

$$a_k|\boldsymbol{\alpha}\rangle = \alpha_k|\boldsymbol{\alpha}\rangle, \tag{11.1.60}$$

then this is a coherent state of the *positive frequency part* electromagnetic field operator;

$$\boldsymbol{E}^{(+)}(\boldsymbol{x},t)|\boldsymbol{\alpha}\rangle = \mathcal{E}(\boldsymbol{x},t)|\boldsymbol{\alpha}\rangle, \tag{11.1.61}$$

where $\mathcal{E}(\boldsymbol{x},t)$ is a function given by

$$\mathcal{E}(\boldsymbol{x},t) = \mathrm{i}\sum_k \left(\frac{\hbar\omega_k}{2\varepsilon_0}\right)^{1/2} \alpha_k \mathbf{u}_k(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\omega_k t}. \tag{11.1.62}$$

The mean value of the electric field in this state is clearly $\mathcal{E}(\boldsymbol{x},t) + \mathcal{E}^*(\boldsymbol{x},t)$.

c) Number State: A number state is one with a definite number of quanta of one mode, thus

$$|k_1, n_1, k_2, n_2, \ldots\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\ldots}{\sqrt{n_1!n_2!\ldots}}|0\rangle. \tag{11.1.63}$$

Such a state is quite hard to create. The mean field in such a state is zero, but the mean square field is non-zero. Number states are, however, the most useful *basis* states for calculations.

Exercise 11.1 Mean Square Electric Field in a Number State: What is the mean square electric field in a number state with n quanta in the mode $k = (\boldsymbol{k}, \lambda)$?

11.2 Coherence and Correlation Functions

The principal difference between quantum mechanics and classical mechanics arises from the wave nature of matter, leading to the importation into mechanics of the characteristically optical phenomena of interference, diffraction, and the concept of coherence as a way of characterizing these phenomena quantitatively. The elementary object of classical wave theory is a wave with a well-defined phase and amplitude, and interference is regarded as arising when two such waves are superposed. Since the phase is considered to be well-defined, interference minima will appear at positions determined by the relative phase between the two waves.

Such ideal wave sources are found essentially only in radio waves, where the phase of the wave can be determined by that of the oscillator at the source. In a laser, the coherent field which emerges has a very stable phase, but it is not determined by any kind of classical oscillator. Thermal sources of light, which are the most common sources, must be treated statistically.

In the remainder of this chapter we will consider the case of scalar fields, such as matter fields and sound fields, and will consider the kinds of correlations and interference that can occur between modes, and thus between the field at different points in space. Time correlations make more sense when there is an underlying dynamical theory which describes how the fields interact with each other or other objects, and these will be dealt with in Chap. 13, and also in *Book II.*

11.2.1 Interference of Classical Waves

Classically, we can imagine fields which are only statistically known, in the sense that the phase and the amplitude are random variables with a certain probability distribution. We can then consider combining together such random fields at points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ by some appropriate interference experiment so that the field at some point $\boldsymbol{r}$ is given by

$$\psi(\boldsymbol{r}) = \psi(\boldsymbol{x}_1) + \psi(\boldsymbol{x}_2). \tag{11.2.1}$$

The mean particle intensity at $\boldsymbol{r}$, where the beams are combined, is

$$I(\boldsymbol{r}) = \langle |\psi(\boldsymbol{r})|^2 \rangle. \tag{11.2.2}$$

Thus total intensity at $\boldsymbol{r}$ is

$$I(\boldsymbol{r}) = \langle |\psi(\boldsymbol{x}_1)|^2 \rangle + \langle |\psi(\boldsymbol{x}_2)|^2 \rangle + 2\,\mathrm{Re}\left\{ \langle \psi^*(\boldsymbol{x}_2)\psi(\boldsymbol{x}_1) \rangle \right\}. \tag{11.2.3}$$

a) First-Order Correlation Function: The quantity

$$G_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \langle \psi^*(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2) \rangle, \tag{11.2.4}$$

is a measure of correlations at different spatial points. It is usual to normalize by dividing by the intensity, so we define the *first-order correlation function* as

$$g_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv \frac{\langle \psi^*(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2) \rangle}{\sqrt{I(\boldsymbol{x}_1) I(\boldsymbol{x}_2)}}, \tag{11.2.5}$$

so that (11.2.3) becomes

$$I(\boldsymbol{r}) = I(\boldsymbol{x}_1) + I(\boldsymbol{x}_2) + 2\sqrt{I(\boldsymbol{x}_1) I(\boldsymbol{x}_2)}\,\mathrm{Re}\left\{ g_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \right\}. \tag{11.2.6}$$

Exercise 11.2 Effect of a Random Phase: The most important effects come from the phases of the fields. Suppose that

$$\psi(\boldsymbol{x}_i) = a(\boldsymbol{x}_i) e^{i\Phi(\boldsymbol{x}_i) + i\delta(\boldsymbol{x}_i)}, \tag{11.2.7}$$

where the $a(\boldsymbol{x}_i)$ are non-random and the $\delta(\boldsymbol{x}_i)$ are mutually Gaussian with mean zero. Show that

$$g_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = e^{-i\Phi(\boldsymbol{x}_1) + i\Phi(\boldsymbol{x}_2)} \exp\left(-\tfrac{1}{2} \left\langle \left(\delta(\boldsymbol{x}_1) - \delta(\boldsymbol{x}_2) \right)^2 \right\rangle \right). \tag{11.2.8}$$

Show that even if the amplitudes $a(\boldsymbol{x}_i)$ are moderately random, the first-order correlation function is not greatly altered.

b) Second-Order Correlation Function: Even when the first-order correlation function is zero, one can obtain interference effects by correlating *intensities*, as was first done by *Hanbury Brown* and *Twiss* [48]. Instead of combining wave sources from two different points and measuring the intensity, one measures the intensity at two different points and correlates these intensities. This defines the *intensity correlation function* or *second-order correlation function*

$$G_2(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \langle |\psi(\boldsymbol{x}_1)|^2 \psi(\boldsymbol{x}_2)|^2\rangle. \tag{11.2.9}$$

A corresponding normalized correlation function is also defined by

$$g_2(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \frac{G_2(\boldsymbol{x}_1,\boldsymbol{x}_2)}{I(\boldsymbol{x}_1)I(\boldsymbol{x}_2)}. \tag{11.2.10}$$

Exercise 11.3 Gaussian Fields: If the fields are jointly Gaussian with no mean field then

$$G_2(\boldsymbol{x}_1,\boldsymbol{x}_2) = I(\boldsymbol{x}_1)I(\boldsymbol{x}_2) + \left|\langle\psi^*(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2)\rangle\right|^2 + \left|\langle\psi(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2)\rangle\right|^2. \tag{11.2.11}$$

In thermal situations, particularly in optics, the correlation function $\langle\psi(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2)\rangle$ vanishes, and the last term is then omitted. In this case we get

$$g_2(\boldsymbol{x}_1,\boldsymbol{x}_2) = 1 + \left|g_1(\boldsymbol{x}_1,\boldsymbol{x}_2)\right|^2. \tag{11.2.12}$$

c) Interference Effects Using Intensity Correlations: Consider a field made by adding two plane waves

$$\psi(\boldsymbol{x}) = r_1 e^{i\phi_1} e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}} + r_2 e^{i\phi_2} e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}}. \tag{11.2.13}$$

The intensity at $\boldsymbol{x}$ is

$$I(x) = r_1^2 + r_2^2 + 2r_1 r_2 \cos\big(\phi_1 - \phi_2 + (\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\boldsymbol{x}\big). \tag{11.2.14}$$

We can now take the intensity correlation function

$$\begin{aligned} G_2(\boldsymbol{x}_1,\boldsymbol{x}_2) = {} & |a_1|^4 + |a_2|^4 + 2|a_1|^2|a_2|^2 \\ & + 2\left(|a_1|^2 + |a_2|^2\right)\mathrm{Re}\left\{a_2^* a_1\left(e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}_1} + e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}_2}\right)\right\} \\ & + 4\,\mathrm{Re}\left\{a_1 a_2^* e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}_1}\right\}\mathrm{Re}\left\{a_1 a_2^* e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}_2}\right\}, \end{aligned} \tag{11.2.15}$$

where

$$a_1 = r_1 e^{i\phi_1}, \qquad a_2 = r_2 e^{i\phi_2}. \tag{11.2.16}$$

Now let us consider the case that the amplitudes a_1 and a_2 have random phases. For convenience we write

$$\phi_1 \to \theta, \qquad \phi_2 \to \theta - \phi, \tag{11.2.17}$$

and average the correlation functions over these phases:

i) The intensity averaged over these angles simply becomes $r_1^2 + r_2^2$, and thus shows no interference fringes.

ii) For the intensity correlation function, the ensemble average of the second line of (11.2.15) vanishes, and the ensemble average of the last line is

$$\frac{1}{4\pi^2}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi\, 4r_1^2 r_2^2 \cos\left(\phi + (\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot \boldsymbol{x}_1\right)\cos\left(-\phi + (\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot \boldsymbol{x}_2\right),$$
$$= 2r_1^2 r_2^2 \cos\left((\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)\right). \tag{11.2.18}$$

Thus, the ensemble average of all of (11.2.15) is

$$G_2(\boldsymbol{x}_1, \boldsymbol{x}_2)|_{\text{Ensemble}} = \left(r_1^2 + r_2^2\right)^2 + 2r_1^2 r_2^2 \cos\left((\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)\right). \tag{11.2.19}$$

iii) This means that we can take two fields with randomized phases, and there will be no interference pattern visible in the intensity, because $G_1(\boldsymbol{x}_1, \boldsymbol{x}_2)$ vanishes, but that nevertheless there is a clearly visible interference pattern in the intensity correlation function. This pattern has the same spatial frequency which would be observed in the intensity if the phases were not completely randomized.

Exercise 11.4 Fringe Visibility: The fringe visibility of the pattern given by (11.2.19), defined by the ratio of maximum to minimum of $G_2(\boldsymbol{x}_1, \boldsymbol{x}_2)|_{\text{Ensemble}}$, is

$$v = \frac{G_2^{\max} - G_2^{\min}}{G_2^{\max} + G_2^{\min}} = \frac{2r_1^2 r_2^2}{\left(r_1^2 + r_2^2\right)^2}. \tag{11.2.20}$$

This has a maximum value of 50%, which happens when $r_1 = r_2$.

d) Physical Interpretation: In practice we imagine the random phasing of the amplitudes to arise as a result of time-averaging over a long period, during which the amplitudes have a definite phase at any given time. Thus, if one did an accurate time-resolved measurement of the intensity, one would see the interference pattern (11.2.14) with time-dependent phases—the pattern would be seen to jitter back and forth, but always remain of the same shape. Thus, the average of this pattern would give the structureless result $r_1^2 + r_2^2$. However the periodic structure of $I(\boldsymbol{x})$ still reveals itself in the intensity correlation function (11.2.19).

11.2.2 Quantum Interference

When considering interference of fields in quantum theory, we need only one *field operator* $\psi(\boldsymbol{x})$, whose modes are populated in such a way as to correspond to interfering modes of the same field. Let us consider therefore a quantum field in which there are only two occupied modes, corresponding to two wavenumbers $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$;

$$\psi(\boldsymbol{x}) = a_1 e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}} + a_2 e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}} + \text{unoccupied modes}. \tag{11.2.21}$$

The correlation functions can be defined in various ways because of choices of operator ordering. The quantum correlation functions conventionally used are *normally ordered*, so that

$$G_1(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \langle \psi^\dagger(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2)\rangle, \tag{11.2.22}$$

$$G_2(\boldsymbol{x}_1,\boldsymbol{x}_2) \equiv \langle \psi^\dagger(\boldsymbol{x}_1)\psi^\dagger(\boldsymbol{x}_2)\psi(\boldsymbol{x}_2)\psi(\boldsymbol{x}_1)\rangle. \tag{11.2.23}$$

Because of the normal ordering, these are zero in the vacuum state.

The *intensity correlation* is defined as the correlation function of the operator intensity

$$I(\boldsymbol{x}) \equiv \psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x}), \tag{11.2.24}$$

and it follows that

$$\langle I(\boldsymbol{x}_1)I(\boldsymbol{x}_2)\rangle = G_2(\boldsymbol{x}_1,\boldsymbol{x}_2) + \delta(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\langle I(\boldsymbol{x}_1)\rangle. \tag{11.2.25}$$

a) Coherent States: The coherent states which are so useful in quantum optics exist for any kind of Boson operators, and are described fully in Chap. 15. Thus, the coherent state can be expressed as

$$|\alpha\rangle \equiv e^{-|\alpha|^2/2}\sum_n^\infty \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{11.2.26}$$

and this leads to the defining equation

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{11.2.27}$$

Such a state is a superposition of states of different particle number, and when these particles are atoms, which are massive and conserved, this cannot represent anything other than a mathematical construction. In contrast, photons can be created and destroyed easily, are not massive, and thus the actual creation of such a state is feasible, and indeed this is a good description of the electromagnetic field of a laser.

b) Interference of Coherent States: If the quantum states of modes 1 and 2 are *coherent states*, written $|\alpha_1,\alpha_2\rangle$, then the density shows interference:

$$\langle\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})\rangle = \langle\alpha_1,\alpha_2|\left(a_1^\dagger e^{-i\boldsymbol{k}_1\cdot\boldsymbol{x}} + a_2^\dagger e^{-i\boldsymbol{k}_2\cdot\boldsymbol{x}}\right)\left(a_1 e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}} + a_2 e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}}\right)|\alpha_1,\alpha_2\rangle, \tag{11.2.28}$$

$$= |\alpha_1|^2 + |\alpha_2|^2 + 2\,\mathrm{Re}\left\{\alpha_2^*\alpha_1 e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}}\right\}. \tag{11.2.29}$$

If we set

$$\alpha_1 \to r_1 e^{i\phi_1}, \qquad \alpha_2 \to r_2 e^{i\phi_2}, \tag{11.2.30}$$

this is identical to the classical result for interference of fields with well-defined amplitude and phase.

Similarly, when we consider the density correlation function for the same coherent states, we get

$$\begin{aligned} G_2(\boldsymbol{x}_1,\boldsymbol{x}_2)|_{\text{Coherent}} = {} & |\alpha_1|^4 + |\alpha_2|^4 + 2|\alpha_1|^2|\alpha_2|^2 \\ & + 2\left(|\alpha_1|^2+|\alpha_2|^2\right)\mathrm{Re}\left\{\alpha_2^*\alpha_1\left(e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}_1} + e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}_2}\right)\right\} \\ & + 4\,\mathrm{Re}\left\{\alpha_1\alpha_2^* e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}_1}\right\}\mathrm{Re}\left\{\alpha_1\alpha_2^* e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}_2}\right\}, \end{aligned} \tag{11.2.31}$$

and making the same replacements (11.2.30), this is also identical to the corresponding classical result (11.2.15).

c) Number States: Let us consider the bivariate number states $|n_1, n_2\rangle$, and use the usual results

$$a_1|n_1, n_2\rangle = \sqrt{n_1}\,|n_1, n_2\rangle, \text{ etc.} \tag{11.2.32}$$

The intensity at the point $\boldsymbol{x}$ becomes

$$\langle n_1, n_2|\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})|n_1, n_2\rangle = n_1 + n_2, \tag{11.2.33}$$

while the correlation function becomes

$$G_2(\boldsymbol{x}_1, \boldsymbol{x}_2)|_{\text{Number}} = \langle a_1^\dagger a_1^\dagger a_1 a_1\rangle + \langle a_1^\dagger a_2^\dagger a_1 a_2\rangle + \langle a_2^\dagger a_1^\dagger a_1 a_2\rangle + \langle a_2^\dagger a_2^\dagger a_2 a_2\rangle$$

$$+ \langle a_2^\dagger a_1^\dagger a_1 a_2\rangle e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)} + \langle a_1^\dagger a_2^\dagger a_1 a_2\rangle e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}, \tag{11.2.34}$$

$$= (n_1 + n_2)^2 + 2n_1 n_2 \cos\big((\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)\big) - n_1 - n_2. \tag{11.2.35}$$

If we set

$$n_1 \to r_1^2, \qquad n_2 \to r_2^2, \tag{11.2.36}$$

the result for the intensity is exactly the same as that for the classical interference of two fields with random phases. The result for G_2 is almost the same, but the quantum result is reduced by a term $n_1 + n_2$. For large occupations n_1, n_2, this becomes negligible, and we can regard the interference between number states as being similar to that from a thermalized classical source.

Exercise 11.5 Fringe Visibility: Show that for the case of number states the visibility is

$$v = \frac{2n_1 n_2}{(n_1 + n_2)^2 - n_1 - n_2}. \tag{11.2.37}$$

When n_1, n_2 are large, this is slightly bigger than the classical result (11.2.20). For the technically difficult, but physically conceivable case of $n_1 = n_2 = 1$

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2) = 2 + 2\cos\big((\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)\big), \tag{11.2.38}$$

and the fringe visibility increases to 100%.

Exercise 11.6 Fermion Interference: If a_1, a_2 are *Fermi* operators, show that

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2) = 2 - 2\cos\big((\boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)\big), \tag{11.2.39}$$

and that fringe visibility is again 100%.

d) Correlated State of Fixed Total Number: Let us consider the projection of the bivariate coherent state $|\alpha_1, \alpha_2\rangle$ onto states of fixed total number of atoms. This is easy to write down directly from the definition (11.2.26) as

$$|M\rangle \equiv \mathrm{e}^{-(|\alpha_1|^2+|\alpha_2|^2)/2} \sum_{n=0}^{M} \frac{\alpha_1^n \alpha_2^{M-n}}{\sqrt{n!(M-n)!}} |n, M-n\rangle. \qquad (11.2.40)$$

Exercise 11.7 Expression in Terms of Coherent States: Show that if

$$\alpha_1 = r_1 \mathrm{e}^{\mathrm{i}\phi_1}, \qquad \alpha_2 = r_2 \mathrm{e}^{\mathrm{i}\phi_2}, \qquad (11.2.41)$$

then we can write

$$|M\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \mathrm{e}^{-\mathrm{i}M\theta} |\alpha_1 \mathrm{e}^{\mathrm{i}\theta}, \alpha_2 \mathrm{e}^{\mathrm{i}\theta}\rangle, \qquad (11.2.42)$$

$$\langle M|M\rangle = \frac{(r_1^2 + r_2^2)^M \mathrm{e}^{-r_1^2 - r_2^2}}{M!}. \qquad (11.2.43)$$

Show that if i, j, k, l take on the values 1,2,

$$\langle M|a_i^\dagger a_j|M\rangle = \frac{M\alpha_i^* \alpha_j}{|\alpha_1|^2 + |\alpha_2|^2}\langle M|M\rangle. \qquad (11.2.44)$$

Similarly

$$\langle M|a_i^\dagger a_j^\dagger a_k a_l|M\rangle = \frac{M(M-1)\alpha_i^* \alpha_j^* \alpha_k \alpha_l}{\left(|\alpha_1|^2 + |\alpha_2|^2\right)^2}\langle M|M\rangle. \qquad (11.2.45)$$

Using the results (11.2.44, 11.2.45) both G_1 and G_2 can be evaluated from the coherent state results (11.2.29, 11.2.31). If $M \gg 1 \Longrightarrow M(M-1) \approx M^2$, these are of the same form as the classical coherent results (11.2.14, 11.2.16) with the correspondences

$$a_1 = \frac{\alpha_1\sqrt{M}}{\sqrt{|\alpha_1|^2 + |\alpha_2|^2}}, \qquad a_2 = \frac{\alpha_2\sqrt{M}}{\sqrt{|\alpha_1|^2 + |\alpha_2|^2}}. \qquad (11.2.46)$$

The Poissonian nature of (11.2.43) means that the dominant contribution to the bivariate coherent state comes from $M \approx |\alpha_1|^2 + |\alpha_2|^2$, so this result is almost indistinguishable from that of the bivariate coherent state.

11.2.3 Summary—Phase and Interference

The classical picture of coherence, correlation and phase sees a wave as having a definite phase, and incoherence as arising as an ensemble or time average over fluctuations of this phase. That is, phase and amplitude exist simultaneously, but may be obscured by fluctuations. The quantum concepts are quite different, but the results for correlation functions and interference are not as different as one might expect. The interference effects found from number states are very similar to those found from classical ensembles with random phases, and the results from interference between correlated states of fixed number are barely distinguishable from either the classical or the results from bivariate coherent states.

The bivariate coherent state representation of interfering beams of bosons is the most convenient one available, but is misleading in a situation where it is clear that no absolute phase exists—the only phase reference for a matter wave is another matter wave. This is in contrast to a classically generated electric field, whose phase is directly related to the phase of the oscillator driving it. Furthermore, the actual strength of the electric field can be measured and its peaks determined; this is truly an absolute phase measurement. For optical fields this is in principle still true, but the frequencies are so high that the direct measurement of the field is not easy. In matter waves the phase we want to talk about is essentially the same as that of a Schrödinger wavefunction, and this truly has no absolute meaning. Nevertheless, it is awkward to do quantum mechanics in such a way as to avoid absolute phases, so we accept them as a convenience which has no physical consequences.

Therefore, from time to time we will choose to use matter wave fields which do have an absolute phase, and this is in particular the case with a Bose–Einstein condensate. If we then take a coherent state $|\alpha_{\boldsymbol{k}}\rangle$ for each mode of the matter wave field, then a mean field arises

$$\Psi(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \alpha_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - i\omega_{\boldsymbol{k}} t)}, \tag{11.2.47}$$

and since this is merely a complex function, it has a phase. However, any actual physics takes place only in subspaces of fixed total number. These are correlated states of fixed total number, that is, multivariate versions of those in Sect. 11.2.2c, but in the case of highly occupied states, such as in a Bose–Einstein condensate, there is no measurable difference between the results predicted by these states and those given by multivariate coherent states.